Finite-size Technology in Low Dimensional Quantum Field Theory, Korea, Yong Pyong, December 2003

Boundary quantum field theories in the Lagrangian framework

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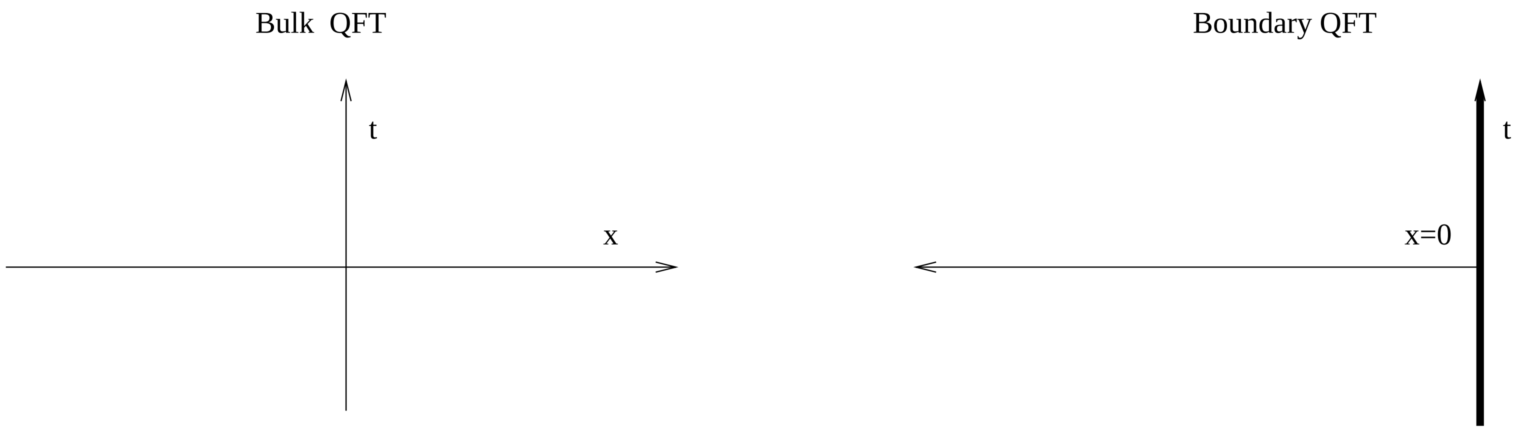


Plan

- Free theory, Asymptotic states, R-matrix, Unitarity

- The analytic structure of the R-matrix, reduction formula, perturbation theory,

- Landau equations, Coleman-Norton interpretation, Cutkosky rules

- Integrable theories: factorization, boundary Yang Baxter equation,

- Bootstrap program

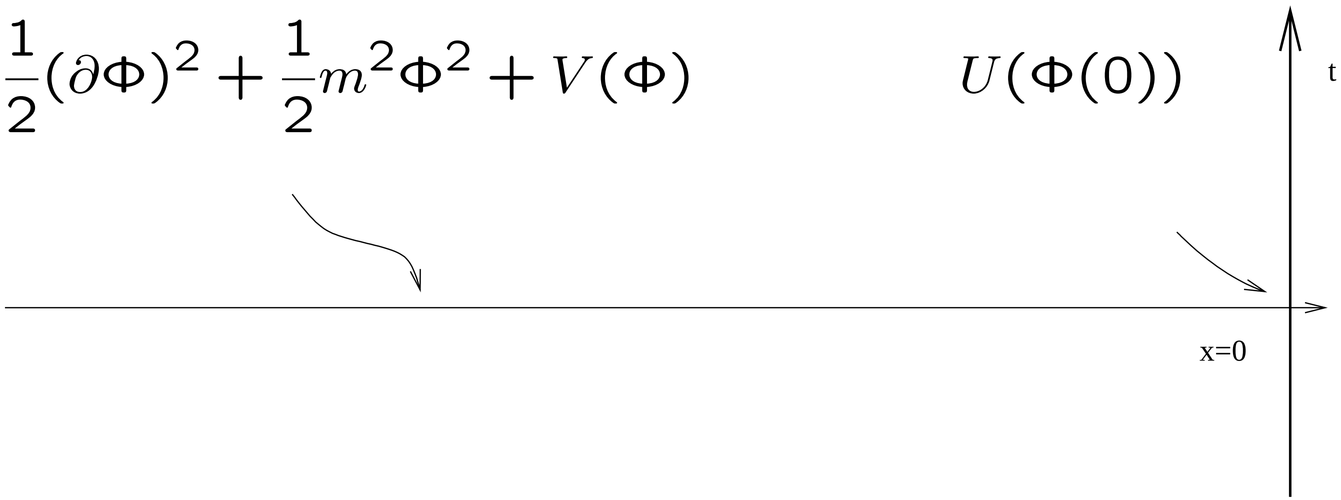
- (Supersymmetric) sine-Gordon model bootstrap program completed

- Checks by finite volume analyzis

Lagrangian descripton of BQFT-s

Lagrangian:

$$\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + V(\Phi) \qquad\qquad U(\Phi(0))$$



Equation of motion, boundary condition

$$\left(\partial_t^2 - \partial_x^2 + m^2\right)\Phi(x,t) = -\frac{\partial V(\Phi)}{\partial \Phi} \qquad {\color{blue}= 0}$$

$$\partial_x \Phi(x,t)|_{x=0} = -\frac{\partial U(\Phi(0,t))}{\partial \Phi(0,t)} \qquad {\color{blue}= 0}$$

Quantization of the free theory

Canonical quantization:

oscillators with frequency $\omega(k) = \sqrt{m^2 + k^2}$

Hilbert space

$$a(k)|0\rangle = 0 \quad ; \quad \forall k$$

$$|k_1, k_2, \ldots, k_n\rangle = a^+(k_1)a^+(k_2)\ldots a^+(k_n)|0\rangle$$

Hamiltonian

$$H = \int_0^\infty dk\omega(k)a^+(k,t)a(k,t)$$

Free propagator

$$\langle T(\Phi(x,t)\Phi(x',t'))\rangle = \int \frac{d^2k}{(2\pi)^2}\frac{ie^{-ik_0(t-t')}}{k^2 - m^2 + i\epsilon}(e^{ik(x-x')} + e^{ik(x+x')})$$

R-matrix

Adiabatical hypothesis: $\mathcal{H}_{in} \equiv \mathcal{H}|_{-\infty} \equiv \mathcal{H} \equiv \mathcal{H}|_{\infty} \equiv H_{out} = H_{free}$

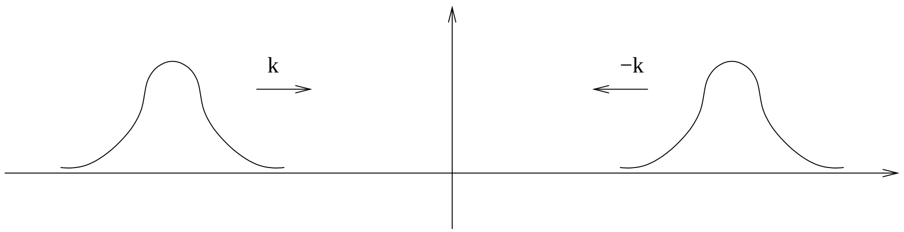
$$|final\rangle_{out} = R|initial\rangle_{in}$$

Simplest physical process

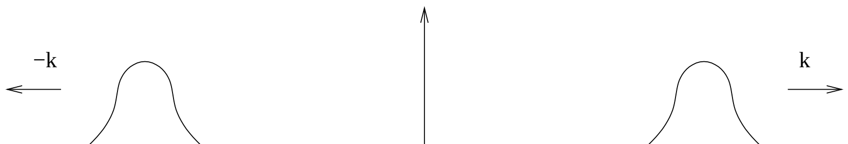
$$|initial\rangle_{in} = \int_{-\infty}^{\infty} d\tilde{k} f(k)|k\rangle_{in}$$

x-dependence: $\tilde{f}(x,t) = \int_{-\infty}^{\infty} d\tilde{k} f(k) \cos(kx) e^{-i\omega t}$

Before reflection



After free reflection



Transition amplitude

Flux

$$i\int dx\tilde{f}^*(x,t)\partial_t\tilde{f}(x,t) = \int d\tilde{k}|f(k)|^2$$

Transition probability into $|final\rangle_{out}$

$$W_{f\leftarrow i} = |\langle final|_{in}R|initial\rangle_{in}|^2$$

Computing the interaction part only

$$R = 1 + iT$$

$$\langle final|T|k\rangle = 2\pi\delta(E(final) - \omega(k))\langle final|\mathcal{T}|k\rangle$$

The connection between a measurable quantity and the matrix element

$$W_{f\leftarrow i} = |f(k(E(final)))|^2|\langle final|\mathcal{T}|k\rangle|^2$$

In the simplest case

$$\langle k'|R|k\rangle = 2\pi(\delta(k-k') + \delta(k+k'))\omega(k)\mathcal{R}(k)$$

Unitarity of the R-matrix

Unitarity equation

$$1 = RR^{+} = (1 + iT)(1 - iT^{+}) = 1 + i(T - T^{+}) + TT^{+}$$

Between one particle states

$${}_{in}\langle k'|1|k\rangle_{in} = {}_{in}\langle k'|RR^{+}|k\rangle_{in}$$

Supposing boundary states

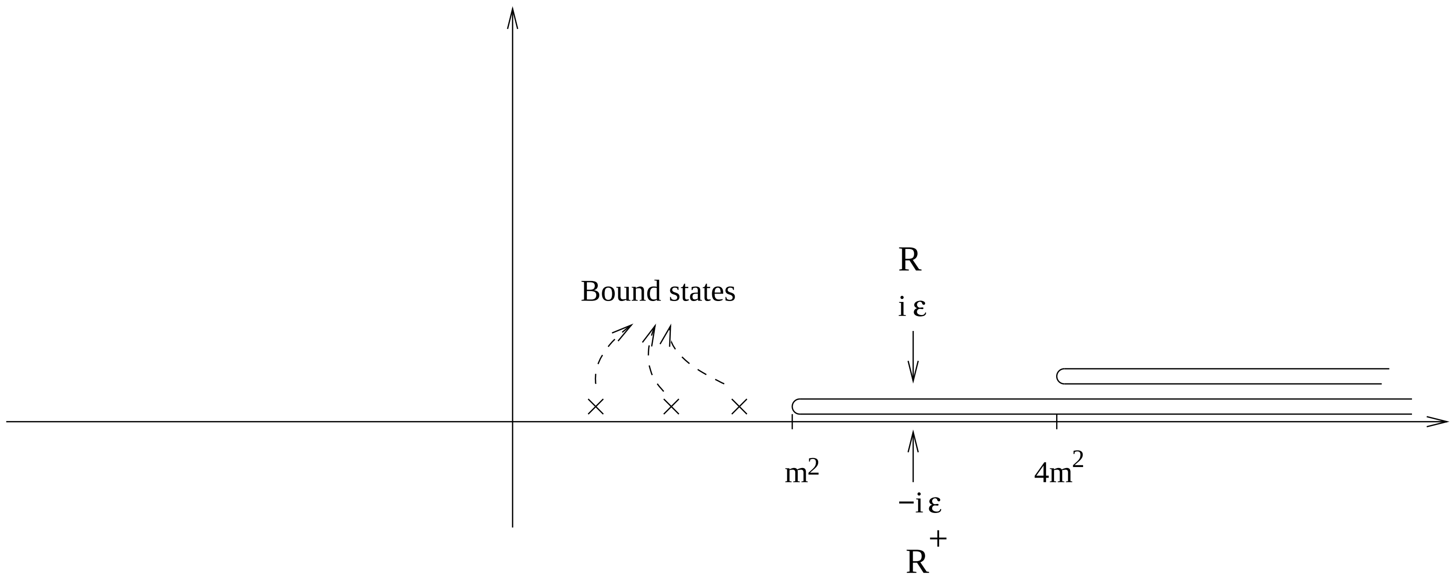
$$= {}_{in}\langle k'|R|B\rangle_{in}\,{}_{in}\langle B|R^{+}|k\rangle_{in} + \int d\tilde{q}\,{}_{in}\langle k'|R|q\rangle_{in}\,{}_{in}\langle q|R^{+}|k\rangle_{in} + \dots$$

Below the two particle threshold

$$\mathcal{R}(k)\mathcal{R}^{*}(k) = 1 \quad \rightarrow \quad \mathcal{R}(k)\mathcal{R}(k^{*}) = 1$$

Discontinuity, normal thresholds: same equations for T

Analitic structure of the R-matrix



Other singularities ?

- Anomalous thresholds
- Boundary crossing unitarity

Reduction formula

Z. B, G. Bohm, G. Takacs: J. Phys.A, hep-th/0207079

$$ {}_{out}< k^{'}|k >_{in} = {}_{out}< k^{'}|a_{in}^{+}(k)|0 >_{in} = $$

Using free fields $\quad a_{in}^{+}(k) = -i2\int_{-\infty}^{0} dx\, \cos(kx) e^{-i\omega(k)t}\partial_t \Phi_{in}(x,t)$

and that $\quad \lim_{t\to-\infty}\Phi(x,t) = Z^{1/2}\Phi_{in}(x,t)$

$$ = {}_{out}< k^{'}|a_{out}^{+}(k)|0 >_{in} + $$

$$ iZ^{-1/2}2\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dt\partial_0\{\cos(kx)e^{-i\omega(k)t}\partial_{0\,out} < k^{'}|\Phi(x,t)|0 >_{in}\} $$

The connected part after partial integration

$$ 2iZ^{-1/2}\int d^2x e^{-i\omega(k)t}\cos(kx)\{\partial_t^2-\partial_x^2+m^2+\delta(x)\partial_x\} < k'|\Phi(x,t)|0 > $$

For the one particle R-matrix

$$ {}_{out}\langle k^{'}|k >_{in} = 2\pi(\delta(k-k') + \delta(k+k'))\mathcal{R}(k) = $$

$$ -4Z^{-1}\int d^2x \int d^2x^{'} \int_{-\infty}^{\infty} dt^{'} e^{i(\omega(k^{'})t^{'}-\omega(k)t)}\cos(kx)\cos(k^{'}x^{'}) $$

$$ \left\{\partial_t^2 - \partial_x^2 + m^2 + \delta(x)\partial_x\right\}\left\{\partial_{t^{'}}^2 - \partial_{x^{'}}^2 + m^2 + \delta(x^{'})\partial_{x^{'}}\right\}\langle 0|T(\Phi(x,t)\Phi(x^{'},t^{'}) $$

Perturbation theory

Compute the Green function

$$G(x,x^{'},t-t^{'}) = \langle 0|T(\Phi(x,t)\Phi(x^{'},t^{'}))|0\rangle$$

Define

$$U(t) = T\exp\left\{-i\int_{-\infty}^{t} dt^{'} H_{int}(t^{'})\right\}$$

Then

$$R = U(\infty) = T\exp\left\{-i\int_{-\infty}^{\infty} dt^{'} H_{int}(t^{'})\right\}$$

and

$$\Phi(x,t) = U^{-1}(t)\Phi_{in}(x,t)U(t)$$

that is

$$G(x,x^{'},t-t^{'}) = \frac{\langle 0|T(\Phi_{in}(x,t)\Phi_{in}(x^{'},t^{'})\exp\left\{i\int d^2x\mathcal{L}_{int}\right\})|0\rangle}{\langle 0|T(\exp\left\{i\int d^2x\mathcal{L}_{int}\right\})|0\rangle} =$$

Perturbative expansion

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}\langle 0|T(\Phi_{in}(x,t)\Phi_{in}(x^{'},t^{'})\int d^2x_1\mathcal{L}_{int}..\int d^2x_n\mathcal{L}_{int})|0\rangle_{Conn.}$$

Feynman rules in coordinate space

For any field associate a dot with coordinate (x, t)

For a term $\alpha\Phi^N$in $V(\Phi)$ associate an N leg vertex with (y, s) and

$$i\alpha \int_{-\infty}^{0} dy \int_{-\infty}^{\infty} ds$$

For a term $\beta\Phi^M$ in $U(\Phi)$ associate an M leg vertex with $(0, s)$ and

$$i\beta \int_{-\infty}^{\infty} ds$$

Between any to dots draw a direct

$$\int \frac{d^2k}{(2\pi)^2} \frac{ie^{-ik_0(t-t')}}{k^2 - m^2 + i\epsilon} (e^{ik(x-x')})$$

or a reflected line

$$\int \frac{d^2k}{(2\pi)^2} \frac{ie^{-ik_0(t-t')}}{k^2 - m^2 + i\epsilon} (e^{ik(x+x')})$$

Feynman rules in momentum space

Momentum space propagator

$$G(x,x^{'},t-t^{'}) = \int \frac{dp}{2\pi} \int \frac{dp^{'}}{2\pi} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t^{'})} e^{ipx} e^{ip^{'}x^{'}} G(p,p^{'},\omega)$$

- direct and reflected inner lines: $\int \frac{dk^2}{(2\pi)^2} \frac{i}{k^2-m^2+i\epsilon}$

- $\beta\Phi^M$ term from $U(\Phi)$:

  boundary vertex with M legs $i\beta 2\pi\delta(\sum k_0)$

- $\alpha\Phi^N$ term from $V(\Phi)$:

  bulk vertex with N legs $i\alpha 2\pi\delta(\sum k_0)\pi\delta(\sum^{'} k_1)$

Landau equations

Z. B., G. Bohm, G. Takacs, Nucl. Phys.B,hep-th/0309119

Generic Feynman graph

$$\int \prod_{i=1}^{L} \frac{d\omega_i}{2\pi} \prod_{j=1}^{K} \frac{dk_j}{2\pi} \prod_{r=1}^{I} (\omega_r^2 - k_r^2 - m^2 + i\epsilon)^{-1}$$

In Feynman parametrization:

$$\int \prod_{i=1}^{L} \frac{d\omega_i}{(2\pi)} \prod_{j=1}^{K} \frac{dk_j}{2\pi} \prod_{r=1}^{I} \int_0^1 d\alpha_i \delta \left(\sum \alpha_i - 1 \right) \left(\sum_{r=1}^{I} \alpha_r (\omega_r^2 - k_r^2 - m^2 + i\epsilon) \right)^{-I}$$

Singularities $\equiv$Landau equations

$$\alpha_r = 0 \quad \text{or} \quad \omega_r^2 - k_r^2 - m^2 = 0 \quad , \quad r = 1, \dots, I$$

$$\frac{\partial}{\partial \omega_i} \left(\sum_{r=1}^{I} \alpha_r (\omega_r^2 - k_r^2 - m^2 + i\epsilon) \right) = 0 \leftrightarrow \sum_{\text{each loop}} \alpha_i \omega_i = 0$$

$$\frac{\partial}{\partial k_j} \left(\sum_{r=1}^{I} \alpha_r (\omega_r^2 - k_r^2 - m^2 + i\epsilon) \right) = 0 \leftrightarrow \sum_{\text{each bloop}} \mu_j \alpha_j k_j = 0$$

Coleman-Norton interpretation, Cutkosky rules

Singularity ↔ Landau equation ↔ existence of a spacetime diagram with particles all on mass shell all moving forward in time such as draw for

direct propagator ↔ vector $\alpha_i(\omega_i, k_i)$ of length $\alpha_i m_i$

reflected propagator ↔ reflected vector $\alpha_i(\omega_i, \pm k_i)$ of length $\alpha_i m_i$

bulk vertex ↔ bulk interaction point with energy-momentum conservation

boundary vertex ↔ boundary interaction point with energy conservation

Discontinuity at the singularity ↔Cutkosky rules

Make the change in the original Feynmal integral

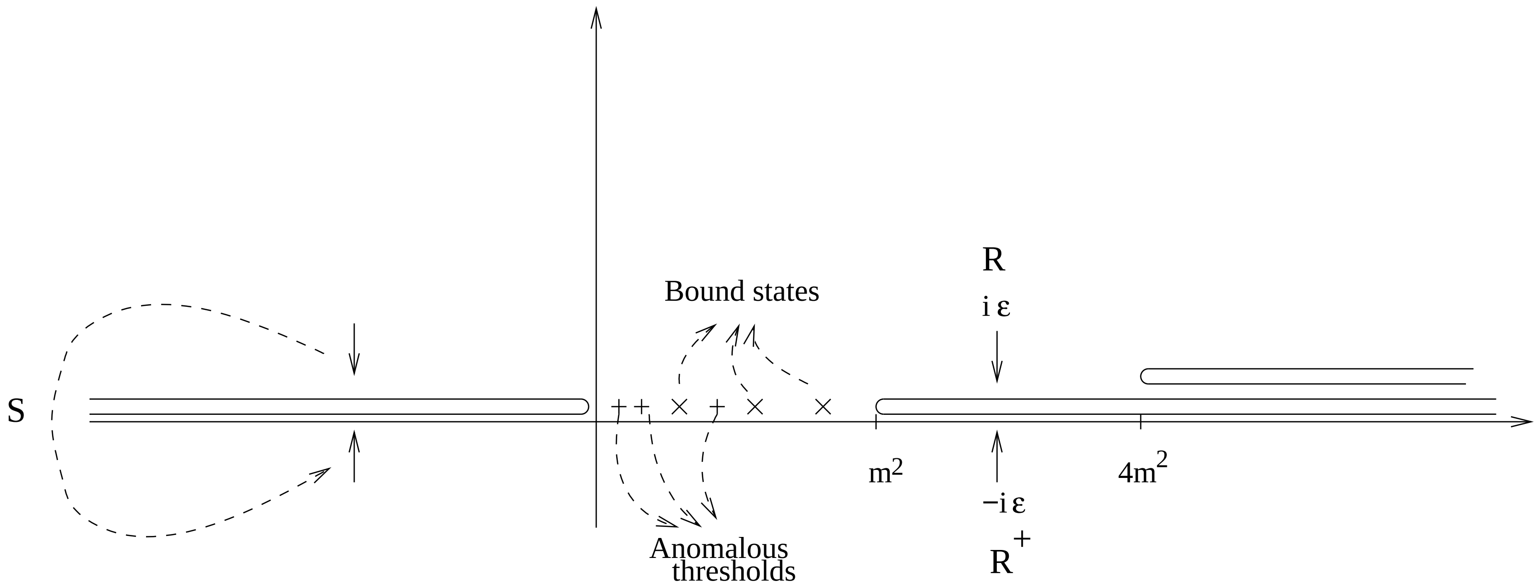
$$(\omega_r^2 - k_r^2 - m^2 + i\epsilon)^{-1} \to \quad -2\pi i\delta^+(\omega_r^2 - k_r^2 - m^2)$$

Coleman-Thun mechanism

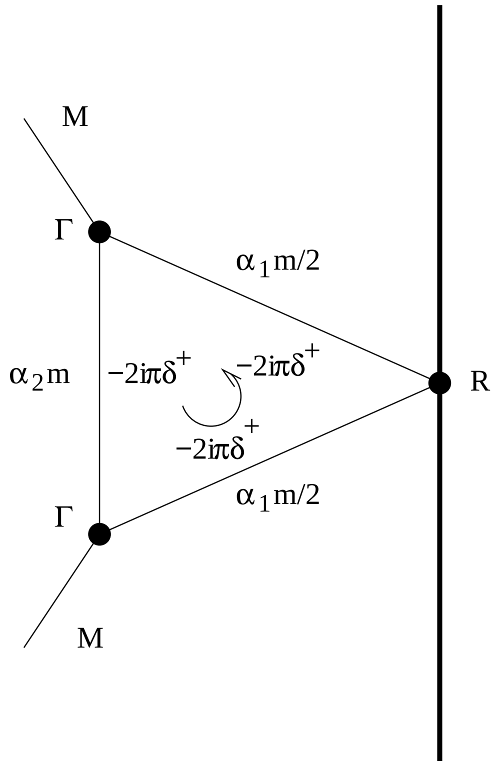
Reduced diagrams: $\alpha = 0$ lines are deleted

Summing up the contributions of the diagrams whose reduced diagrams are the same $\leftrightarrow$ Coleman-Thun rules: The actual three level couplings has to be replaced by the exact vertex functions

The total contribution: poles in the R-matrix



Example for the Coleman-Thun mechanism



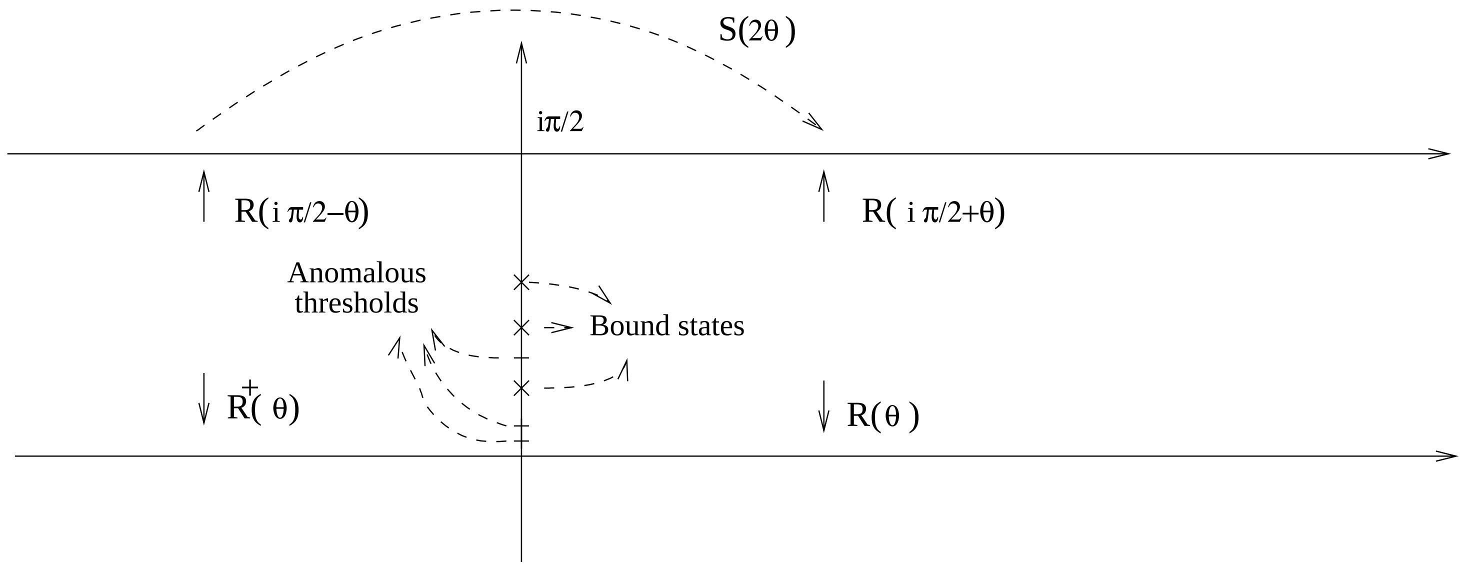
Expressing in terms of the physical quantities

$$\text{res} R_M(iu) = -\frac{1}{2} R_m(2iu)) \,\text{res} S(2iu)$$

Integrable models

Infinitely many conserved charges $\leftrightarrow$ No particle production

Introduce $\omega = m \cosh \theta$



Unitarity:

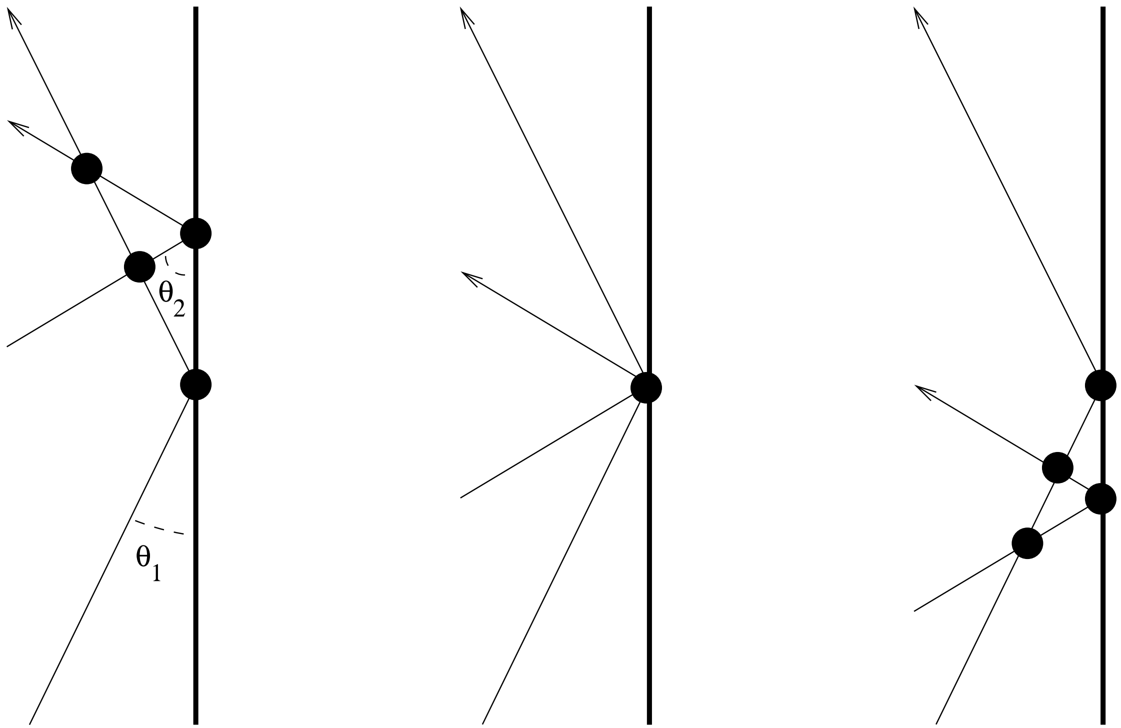
$$R(\theta)R(-\theta) = 1$$

Boundary crossing unitarity S. Ghoshal, A. Zamolodchikov: IJMPA9 (1994) 3841, 4801.

$$R(\frac{i\pi}{2} - \theta) = S(2\theta)R(\frac{i\pi}{2} + \theta)$$

Boundary Yang Baxter Equation

Higher spin conserved charges ↔ Trajectories can be shifted ↔ Factorization

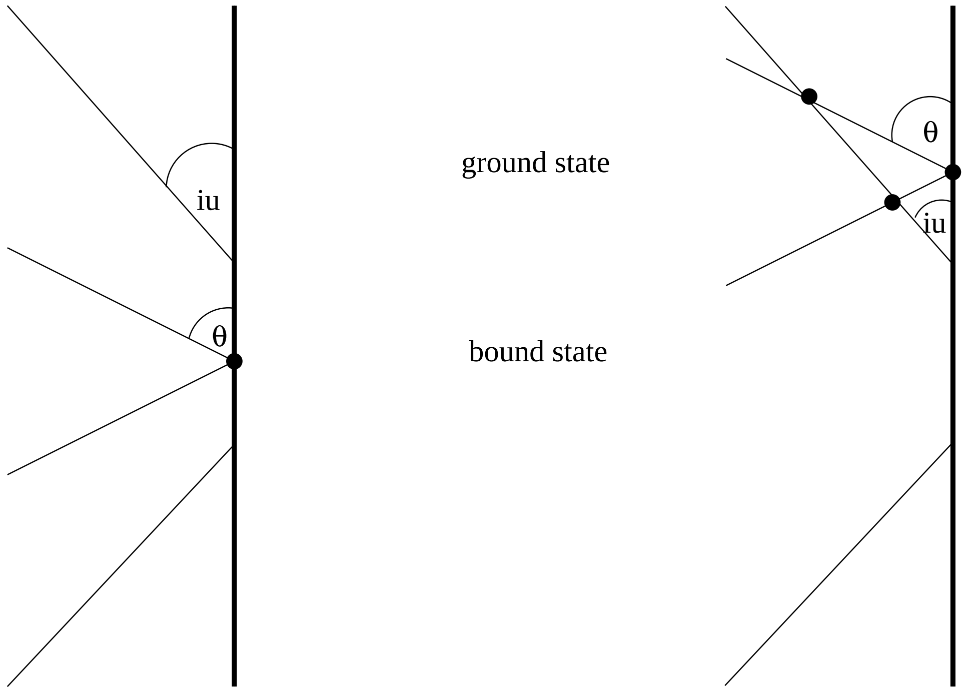


Boundary Yang-Baxter (matrix) equation I. Cherednik, Theor. Math. Phys. 61,35,997 (1984)

$$R(\theta_1)S(\theta_1+\theta_2)R(\theta_2)S(\theta_1-\theta_2) = S(\theta_1-\theta_2)R(\theta_2)S(\theta_1+\theta_2)R(\theta_1)$$

Reflection factors on excited states: boundary bootstrap

Boundary bootstrap, A. Fring, R. Koberle: Nucl.Phys.B421:159-172,1994



$$R_{boundstate}(\theta) = R_{groundstate}(\theta)S(\theta + iu)S(\theta - iu)$$

Bootstrap program

1. Take a solved bulk model with nontrivial BYBE and solve the BYBE

2. Find Coleman-Thun explanation of poles for $0 < \Im m(\theta) < \pi$

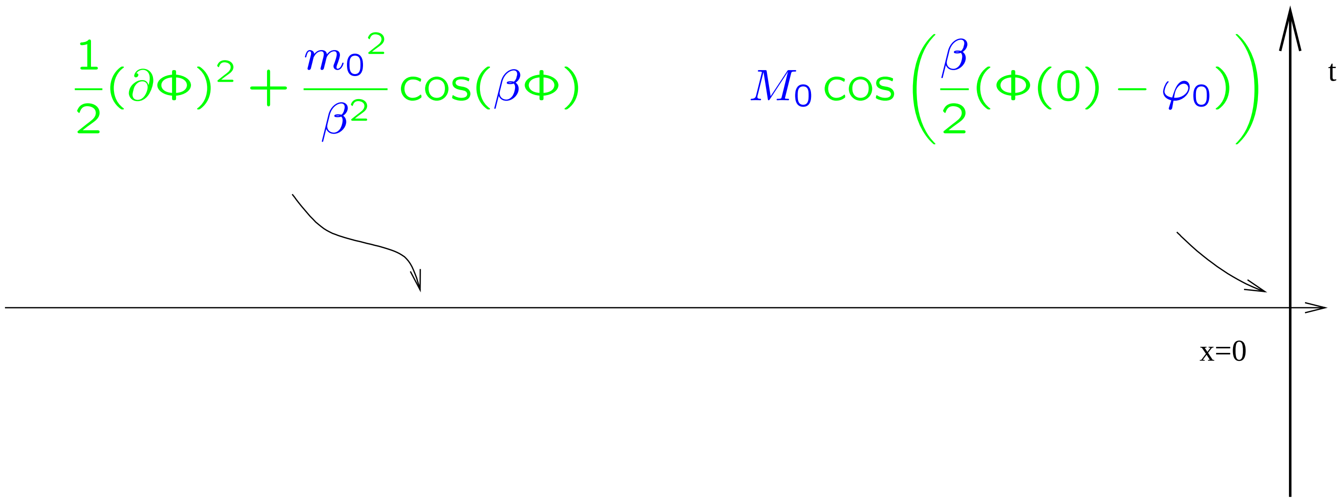
3. For poles without CT associate boundary excited states

4. Compute the reflection matrix on the excited boundary states

5. Analyze the pole structure of the excited R-matrix

6. The program is completed if at the end only CT poles remain

Boundary sine-Gordon model



Short boundary history:

Integrability, ground state reflection factors: S. Ghoshal, A. Zamolodchikov: IJMPA9 (1994) 3841, 4801.

Partial Dirichlet ($M_0 \to \infty$) spectrum (no Coleman-Thun): S. Skorik, H. Saleur: JPA28 (1995) 6605.

UV-IR relation: Al. B. Zamolodchikov (unpublished).

General Dirichlet ($M_0 \to \infty$) spectrum (with Coleman-Thun): P. Mattsson, P. Dorey: JPA33 (2000) 9065.

Neumann spectrum: Z.B., L. Palla, G. Takács: NPB614 (2001) 405.

General spectrum, TCSA, TBA verification: Z.B., L. Palla, G. Takács, G.Zs.Tóth: NPB622 (2002) 548, 565.

Semi-classical issues: L. Palla, M. Kormos:J.Phys. A35 (2002) 5471-5488.

TBA in reflectionless points: T. Lee, Ch. Rim: Nucl.Phys. B672 (2003) 487, J.-S. Caux, H. Saleur, F. Siano:Nucl.Phys. B672 (2003) 411.

Bulk data

particle spectrum: soliton and antisoliton with S-matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{-\sin\lambda\pi}{\sin\lambda(\pi+i\theta)} & \frac{\sin i\lambda\theta}{\sin\lambda(\pi+i\theta)} & 0 \\ 0 & \frac{\sin i\lambda\theta}{\sin\lambda(\pi+i\theta)} & \frac{-\sin\lambda\pi}{\sin\lambda(\pi+i\theta)} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \prod_{l=1}^{\infty} \left[\frac{\Gamma(2(l-1)\lambda+\frac{\lambda i\theta}{\pi})\Gamma(2l\lambda+1+\frac{\lambda i\theta}{\pi})}{\Gamma((2l-1)\lambda+\frac{\lambda i\theta}{\pi})\Gamma((2l-1)\lambda+1+\frac{\lambda i\theta}{\pi})} / (\theta \to -\theta\right.$$

Bound states: breathers

$$m_{B^n} = 2M \sin\frac{n\pi}{2\lambda}$$

Scattering on the solitons

$$S^n = \{n-1+\lambda\}\{n-3+\lambda\}\dots$$

Scattering amoung themselves

$$S^{n,m} = \{n+m-1\}\{n+m-3\}\dots\{n-m+3\}\{n-m+1\}$$

$$\{y\} = \frac{\left(\frac{y+1}{2\lambda}\right)\left(\frac{y-1}{2\lambda}\right)}{\left(\frac{y+1}{2\lambda}-1\right)\left(\frac{y-1}{2\lambda}+1\right)} \qquad (x) = -\frac{\sin(x\pi/2-i\theta/2)}{\sin(x\pi/2+i\theta/2)}$$

UV-IR relation

$$\lambda = \frac{8\pi}{\beta^2}-1 \quad ; \qquad M = m_0^{\frac{8\pi}{8\pi-\beta^2}}\kappa(\beta)$$

1. Solution of the BYBE

Integrability→no particle production and the R-matrix factorizes into the product of two particle S-matrices and one particle R-matrices

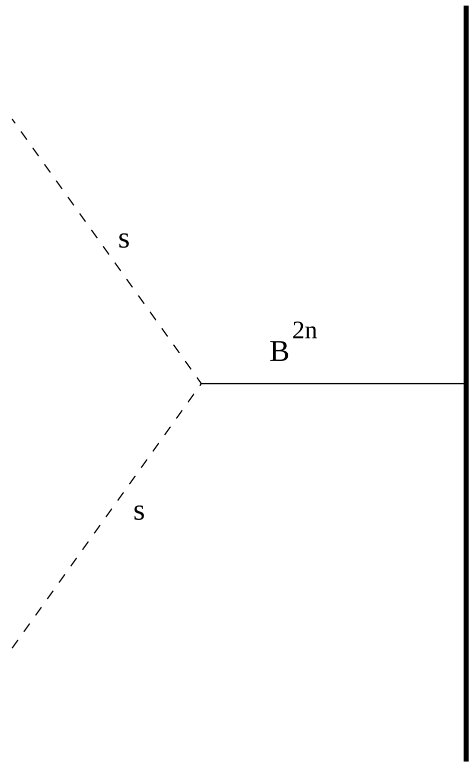
Constraints: boundary Yang-Baxter equation, unitarity, boundary crossing symmetry. The most general solution

$$R(\lambda,\eta,\Theta) = \begin{pmatrix} P^{+} & Q \\ Q & P^{-} \end{pmatrix} R_0(\theta)\frac{\sigma(\eta,\theta)}{\cos\eta}\frac{\sigma(i\Theta,\theta)}{\cosh\Theta}$$

$$P^{\pm}{=}\cos(i\lambda\theta)\cos\eta\cosh\Theta\pm(\cos\leftrightarrow\sin) \quad ; \quad Q{=}\cos i\lambda\theta\sin i\lambda\theta$$

2. Coleman-Thun poles

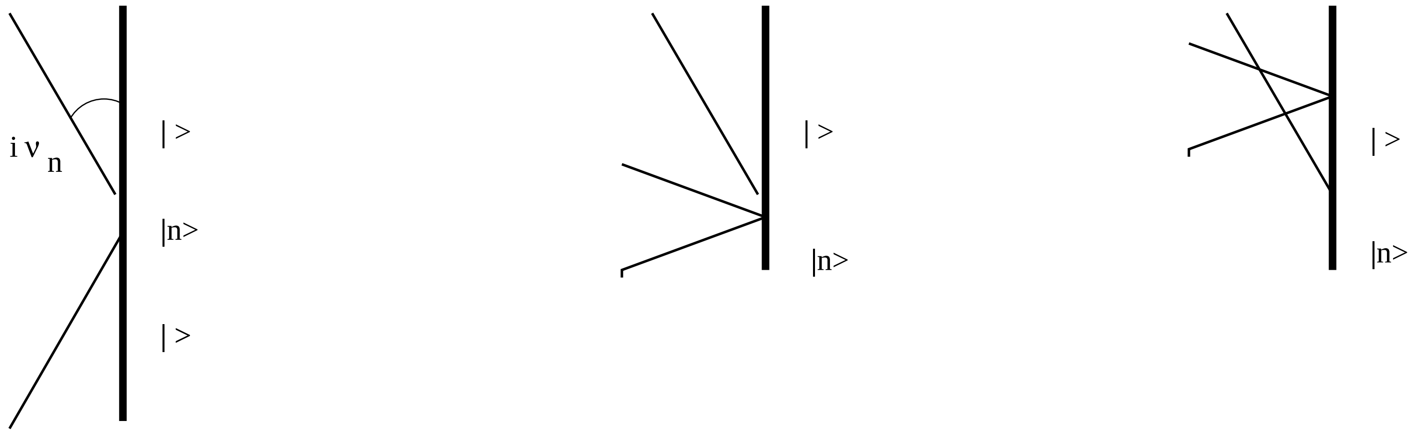
Boundary independent CT poles in $R_0(\theta)$



3-4. Boundary excited states, reflection factors

Boundary dependent poles in $\sigma(\eta, \theta)$

poles at $\theta = i\nu_n = \left(\frac{\eta}{\lambda} - \frac{(2n+1)}{2\lambda}\right)$

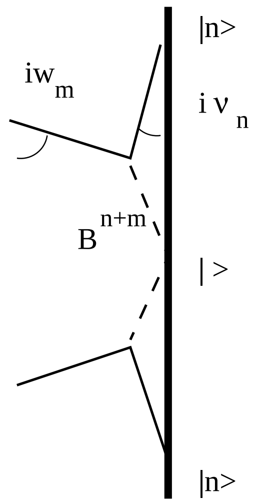


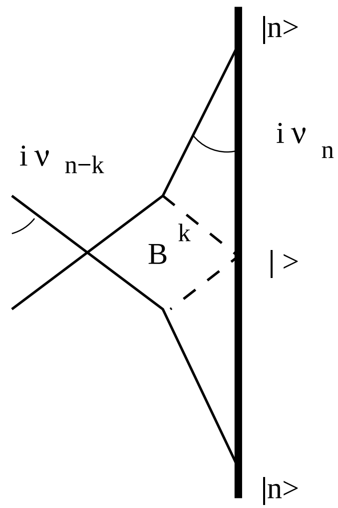
$$m_{|n\rangle} = M \cos(\nu_n) \quad ; \quad \begin{array}{c} R_{|n\rangle}(\lambda, \eta, \Theta) = \bar{R}(\lambda, \bar{\eta}, \Theta) a_n(\eta, \theta) \\ \bar{R} = R^{s \leftrightarrow \bar{s}} \; ; \; \bar{\eta} = \pi(\lambda + 1) - \eta \end{array}$$

5. Pole analysis, excited wall

poles at

$\theta = iw_m = i\nu_m(\bar{\eta})$ $\theta = i\nu_{n-k}$





exists only for $w_m < \nu_n$ for $w_m > \nu_n$ new boundary boundstate $|n, m\rangle$: $m_{|n,m\rangle} = M(\cos(\nu_n) + \cos(w_m))$ if $w_m < \frac{\pi}{2}$

Boundary spectrum

$|\,\rangle \quad R(\lambda,\eta,\Theta)$

$|0\rangle\ \bar{R}(\lambda,\bar{\eta},\Theta)$ $\quad\ldots\quad$ $|n\rangle\ \bar{R}(\lambda,\bar{\eta},\Theta)a_n(\eta,\theta)$

$M\cos(\nu_0)$ $\qquad$ $M\cos(\nu_n)$

$|n,m\rangle\ R(\lambda,\eta,\Theta)a_n(\eta,\theta)a_m(\bar{\eta},\theta)$

$M\cos(\nu_n)+M\cos(w_m)$

$|n_1,m_1,\ldots,n_k\rangle\ \bar{R}(\lambda,\bar{\eta},\Theta)a_{n_1}(\eta,\theta)a_{m_1}(\bar{\eta},\theta)\ldots a_{n_k}(\eta,\theta)$

$M\cos(\nu_{n_1})+M\cos(w_{m_1})+\ldots+M\cos(\nu_{n_k})$

$|n_1,m_1,\ldots,m_k\rangle\ R(\lambda,\eta,\Theta)a_{n_1}(\eta,\theta)a_{m_1}(\bar{\eta},\theta)\ldots a_{m_k}(\bar{\eta},\theta)$

$M\cos(\nu_{n_1})+M\cos(w_{m_1})+\ldots+M\cos(w_{m_k})$

Finite volume analysis, fix points

UV limit

c=1 BCFT $r\beta = \sqrt{4\pi}$

$$H = \frac{1}{8\pi}\int_0^L \left((\Pi)^2 + (\partial_x \Phi)^2\right) dx$$

Spectrum

$$a_{-n_1}\dots a_{-n_k}|n\rangle \ ; \ \Pi_0|n\rangle = \frac{n}{r}|n\rangle$$

$$H = \frac{\pi}{L}\left(2\Pi_0^2 + \sum_{n\neq 0} n a_{-n} a_n\right)$$

IR limit

Bulk spectrum

$$s, \bar{s}, B^1 \dots B^n$$

$$S(\lambda)\dots$$

Boundary spectrum

$$|n_1, m_1, \dots, m_k\rangle$$

$$R(\lambda, \eta, \Theta)\dots$$

UV fix point

L=0

IR fix point

L=infinite

Finite volume analysis, near the fix points

Near UV: TCSA

Scaling fields

$$V_n(x,t) =: e^{i\frac{n}{r}\Phi(x,t)} :$$

$$\Psi_n(0,t) =: e^{i\frac{n}{r}\Phi(0,t)} :$$

$$H^{pert.}_{bulk} \to \frac{1}{2}(V_2 + V_{-2})$$

$$H^{pert.}_{bd.} \to \frac{1}{2}(e^{-\frac{i}{r}\varphi_0}\Psi_1 + e^{\frac{i}{r}\varphi_0}\Psi_{-1})$$

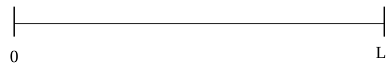
Diagonalize $H^{pert}(m_0, M_0)$

Near IR: BY lines

Momentum quantization

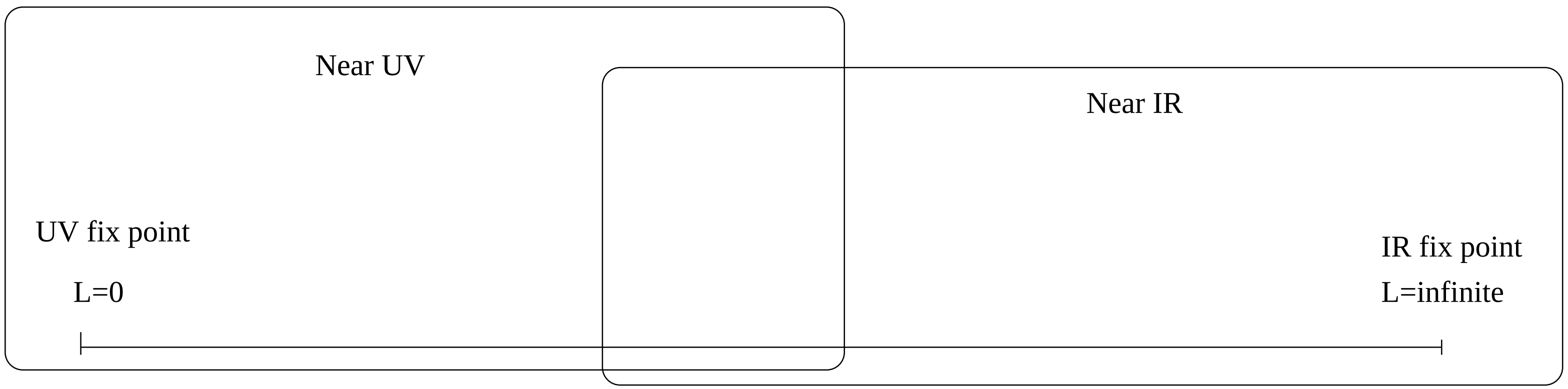
Bulk: $e^{iPL} = 1\,,\, P(L) = \frac{2\pi}{L}N$

Boundary

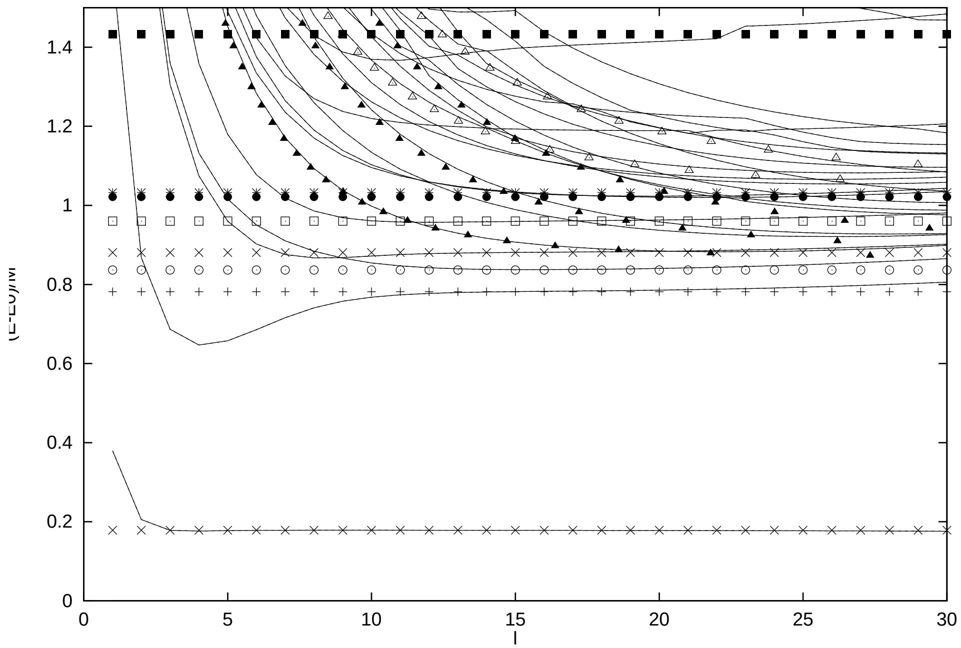


$$e^{i2PL}R_0(P)R_L(P) = 1 \to P(L)$$

$$E(L) = \sqrt{M^2 + P(L)^2}$$



Numerical test



Exact calculations

VEV of V_n $(m_0, M_0, \beta, \varphi_0) \rightarrow$ $E_{bound.}$ $\leftarrow$TBA equation $(M, \lambda, \eta, \Theta)$

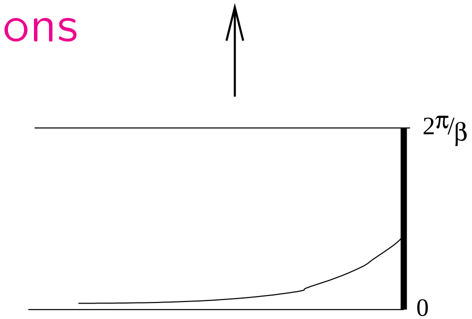
Semi-classical issues

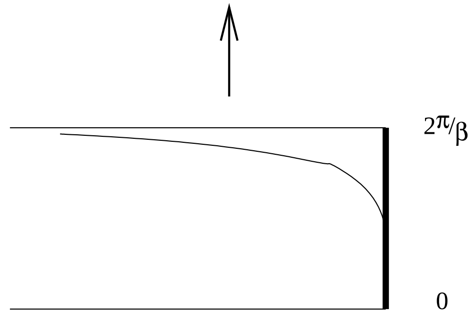
Boundary boundstates related by $s \leftrightarrow \bar{s}, \eta \leftrightarrow \bar{\eta}$

$|\,\rangle \quad R(\lambda, \eta, \Theta) \quad ; \quad |0\rangle\ \bar{R}(\lambda, \bar{\eta}, \Theta)$

$M\cos(\nu_0)$

Classical static solutions





Semi-classical corrections: linearized fluctuations

Discrete spectrum:

nothing $\qquad \omega_0^2 > 0$

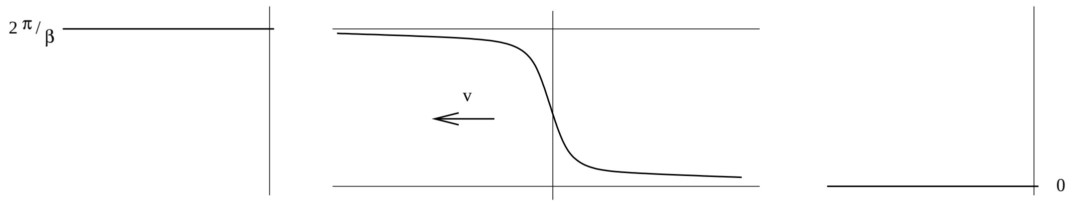
Energy differences $\longrightarrow$ semi-classical UV-IR relation

$$\Delta E_{classical} + \Delta E_{semi-classical} \leftrightarrow M\cos(\nu_0)$$

$$\omega_0 \leftrightarrow M\cos(\nu_1) - M\cos(\nu_0)$$

Instability for $\varphi_0 = 0$

Discrete spectrum: $\omega_0^2 < 0$



resonance pole in $\sigma(i\Theta, \theta)$ at $\theta = -\nu_0 = -\frac{\Theta}{\lambda} - i\frac{\pi}{2\lambda}$

with the same energy and width in the classical limit.

Reflection factors, unstable boundstate are confirmed in time delay analysis as well

Boundary supersymmetric sinh-Gordon model

History:

Boundary

Lagrangian: T. Inami , S. Odake, Y-Z. Zhang, Phys.Lett.B359:118-124,1995.

Boundary reflection matrices: Ch. Ahn , W. M. Koo: J.Phys.A29:5845-5854,1996.

Perturbative checks for sinh-Gordon: M. Ablikim, E. Corrigan: Int.J.Mod.Phys. A16 (2001) 625

Lagrangean with two parameters: R. I. Nepomechie: Phys.Lett. B509:183-188,2001.

Bootstrap closed: Z. B, L. Palla, G. Takacs: NLIE : C. Dunning: Nucl.Phys. B644 (2002) 509-532.

Bulk in finite volume

NLIE : C. Dunning:J.Phys.A36:5463-5476,2003

CPT,NLIE,TCSA : Z. B., C. Dunning, L. Palla, G. Takacs, F. Wagner: Nucl. Phys. B. hep-th/0309120.