Higher Spin AdS Duality from CFT: a constructive approach

Antal Jevicki
Brown University
June 10, 2011

OVERVIEW

- Large N expansion implemented through the AdS/CFT correspondence is at present our main tool for understanding the gauge/string duality
- *N*=4 SYM: gives exact results summarized by the Bethe ansatz
- g=0: Tensionless string [Sundborg ‘94; Witten ‘01; Sezgin & Sundell ‘02]
- Vector models, at critical points, result in AdS_4/CFT_3 correspondence [Klebanov & Polyakov ‘02] with pure AdS spacetime

- As a result, the duality is not restricted to Gauge/String theories
- Higher Spin Gravity [Vasiliev ’80-’90]
- Agreement of boundary correlation functions:

  This has been demonstrated by Giombi and Yin [’09, ‘10]
- The model (free CFT) gives us the simplest example of AdS/CFT correspondence: in depth study

Talk:

- Present a <u>direct</u> construction of AdS_4 HS gravity from field theory

  S. Das, AJ '03; R. de Mello Koch, K, Jin, AJ, J. Rodrigues '10

- The construction is based on the notion of collective fields

  conformal operators $\{O_I\} \subset \{\Phi_C\}$ collective fields: Bi-local

- Represents a direct change of variables

$$Z = \int \prod_{i=1}^{N} d\phi^i(x) e^{-S[\phi]} = \int d\Phi_c e^{-\kappa \bar{S}[\Phi_c]} \qquad \Phi_c = \phi(x) \cdot \phi(y)$$

$\kappa = N$: vector model $\qquad \kappa = N^2$: matrix model

- $\bar{S}[\Phi_c]$ an effective action for Φ_c can be explicitly constructed and will be used to give a complete description of AdS space-time (with the extra dimension and interactions)

- This construction provides a theoretical laboratory for further studies of AdS/CFT:

1. Origin of the extra AdS dimension
2. Locality of emerging AdS space-time
3. Bulk interactions in AdS space-time
4. Black hole solutions and formation
5. Finite N cutoff: exclusion principle

- Re: Origin of the extra dimension

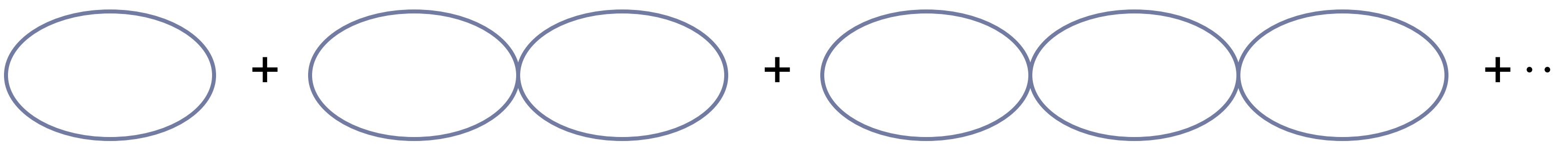
1. Holographic framework: in AdS/CFT it is used to motivate the procedure of using CFT data (correlation functions) at the z=0 boundary of AdS space-time
2. The direct (collective) field construction for reaching the points and interactions in the bulk should be compared with: Holographic renormalization group [Douglas, Mazzucato, Razamat; S.S. Lee; H. Liu]

I. CFT_3: the vector model

- Vector model in 3 dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi}\cdot\partial\vec{\phi} + \frac{1}{2}m^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2$$

- Large N: bubble diagrams



$N \to \infty$ λN - fixed

- Two conformal (critical) theories:
  - Free theory: $\lambda = 0$ UV
  - Interacting: $\lambda \neq 0$ IR

- Connected two-point functions

$$\langle\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\rangle = \frac{\delta(k_1+k_2)}{8|k_1|}\left[1-\frac{1}{8|k_1|}\frac{2\lambda}{1+\frac{2\lambda}{8|k_1|}}\right] = \frac{\delta(k_1+k_2)}{\lambda+8|k_1|}$$

- UV fixed point: conformal dimension=1

$$\langle\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\rangle = \frac{\delta(k_1+k_2)}{8|k_1|}$$

- IR fixed point: conformal dimension=2

$$\langle\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\rangle = \delta(k_1+k_2)\left[\frac{1}{\lambda_0\Lambda}-\frac{8|k_1|}{\lambda_0\Lambda^2}+\mathcal{O}\left(\frac{|k_1|^2}{\Lambda^3}\right)\right]$$

Critical point ➡ conserved currents:

- Free theory:

Makeenko '81; Mikhailov '02
Braun, Korchemsky, Muller '03

$$J_{\mu_1\ldots\mu_s} = \sum_{k=0}^{s} (-1)^k \binom{s-1/2}{k} \binom{s-1/2}{s-k} \partial_{\mu_1} \cdots \partial_{\mu_k}\phi \, \partial_{\mu_{k+1}} \cdots \partial_{\mu_s}\phi - \text{traces}$$

can be packed into a generating function

$$\mathcal{O}(\vec{x},\vec{\epsilon}) = \phi^i(x-\epsilon) \sum_{n=0}^{\infty} \frac{1}{(2n)!} (2\epsilon^2 \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_x - 4(\epsilon \cdot \overleftarrow{\partial}_x)(\epsilon \cdot \overrightarrow{\partial}_x))^n \phi^i(x+\epsilon)$$

where $\epsilon^2 = 0$ corresponds to traceless condition.

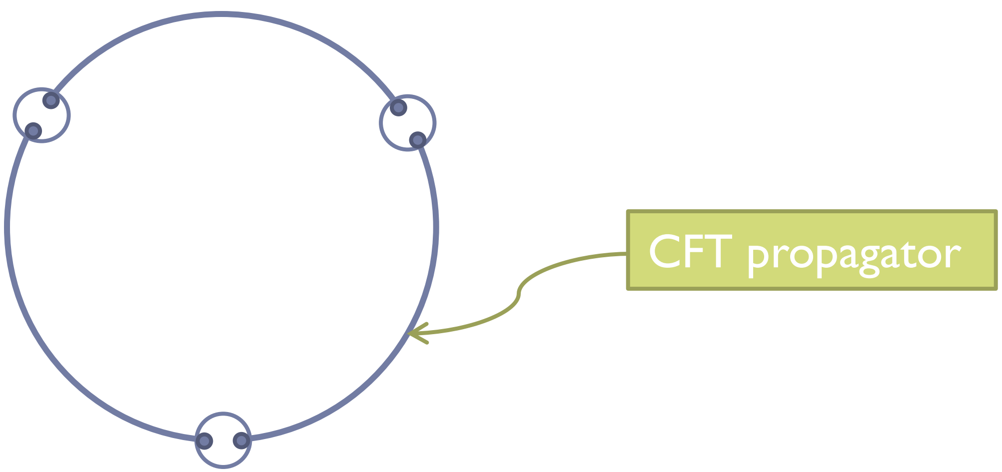
Coupling to Higher-Spin: Bekaert

Three-point function:

- Giombi-Yin calculated three-point function

$$\langle \mathcal{O}_{\epsilon_1}(x_1)\mathcal{O}_{\epsilon_2}(x_2)\mathcal{O}_{\epsilon_3}(x_3)\rangle$$

corresponds to the diagram:



Exact agreement with z=0 Vasiliev's theory amplitudes.

Collective dipole representation:

- Exact construction

- Change from field $\vec{\phi}(x) = (\phi_1, \phi_2, \ldots \phi_N)$ to the bilocal field:

$$\Phi(x, y) = \phi(x) \cdot \phi(y) = \sum_{a=1}^{N} \phi^a(x)\phi^a(y)$$

O(N) invariant

3d + 3d

- Represents a more general set than the conformal fields:

$$\mathcal{O}(x, \epsilon) \qquad \epsilon^2 = 0$$

3d + 2d

The collective (effective) action:

- Partition function:

$$Z = \int [d\phi^i(x)] e^{-S[\phi]} = \int \prod_{x,y} d\Phi(x,y)\mu(\Phi) e^{-S_c[\Phi]}$$

- The collective(effective) action:

$$S_{eff} = \mathrm{Tr}[-(\partial_x^2 + \partial_y^2)\Phi(x,y) + m^2\Phi(x,y) + V] + \frac{N}{2}\mathrm{Tr}\ln\Phi$$

- Origin of the $\ln\Phi$ interaction: Jacobian

$$\int d\vec{\phi} e^{-S} \to \int d\Phi \det\left|\frac{\partial\phi^a(x)}{\partial\Phi(x_1, x_2)}\right| e^{-S}$$

- The measure:

$$\mu(\Phi) = (\det\Phi)^{V_x V_p}$$

$V_x = L^3$ space

$V_p = \Lambda^3$ momentum cutoff

- Definition of trace

$$\mathrm{Tr}B = \int d^3x B(x,x)$$

- Star product

$$(\Psi * \Phi)(x,y) = \int dz \Psi(x,z)\Phi(z,y)$$

- Tr and Det are natural in bi-local theory.

Properties of the collective field reformulation:

- Same partition function
- Collective field is more general than the conformal fields

$$\mathcal{O}(\vec{x}, \vec{\epsilon}) \subset \Phi(\vec{x}, \vec{y})$$

3 + 2 3 + 3

➡ Bulk (AdS_4) representation

- AdS_4 HS gravity coupling constant:

$$g = \frac{1}{\sqrt{N}}$$

Expansion

- Equation of motion:

$$\frac{\partial S_c}{\partial \Phi} = 0 \Leftrightarrow -(\partial_x^2 + \partial_y^2)\delta(x-y) + \frac{\lambda}{2}\Phi(x,y) + N\frac{1}{\Phi(x,y)} = 0$$

- 1/N expansion parameter:

$$\Phi = \Phi_0 + \frac{1}{\sqrt{N}}\eta$$

$$S_c = S[\Phi_0] + \mathrm{Tr}[\Phi_0^{-1}\eta\Phi_0^{-1}\eta] + \frac{g}{4}\eta\eta$$

$$+\frac{1}{\sqrt{N}}\mathrm{Tr}[\Phi_0^{-1}\eta\Phi_0^{-1}\eta\Phi_0^{-1}\eta] + \sum_n N\Big(\frac{1}{\sqrt{N}}\Big)^n \mathrm{Tr}(B^n)$$

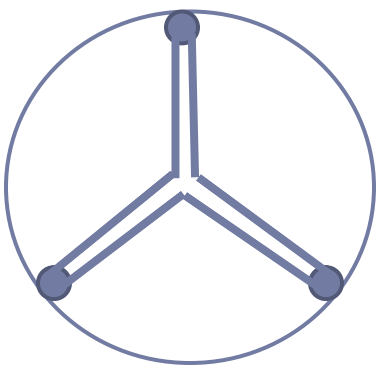
$$B = \Phi_0^{-1}\eta$$

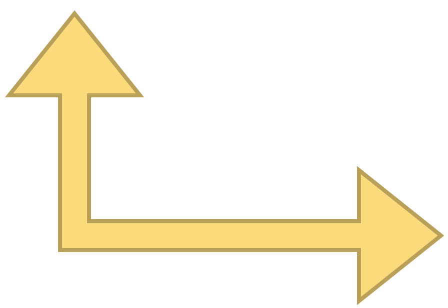
- S_c-exact: Reproduces all invariant correlators

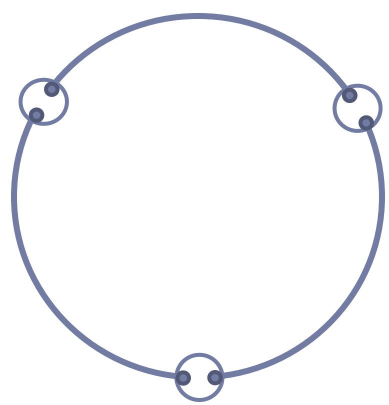
$$\langle \phi(x_1)\cdot\phi(y_1)\ \phi(x_2)\cdot\phi(y_2)\cdots\phi(x_n)\cdot\phi(y_n)\rangle$$

Topology: Witten diagram! (bulk AdS)

1  $\langle \Phi(x_1, y_1)\Phi(x_2, y_2)\rangle$ Propagator

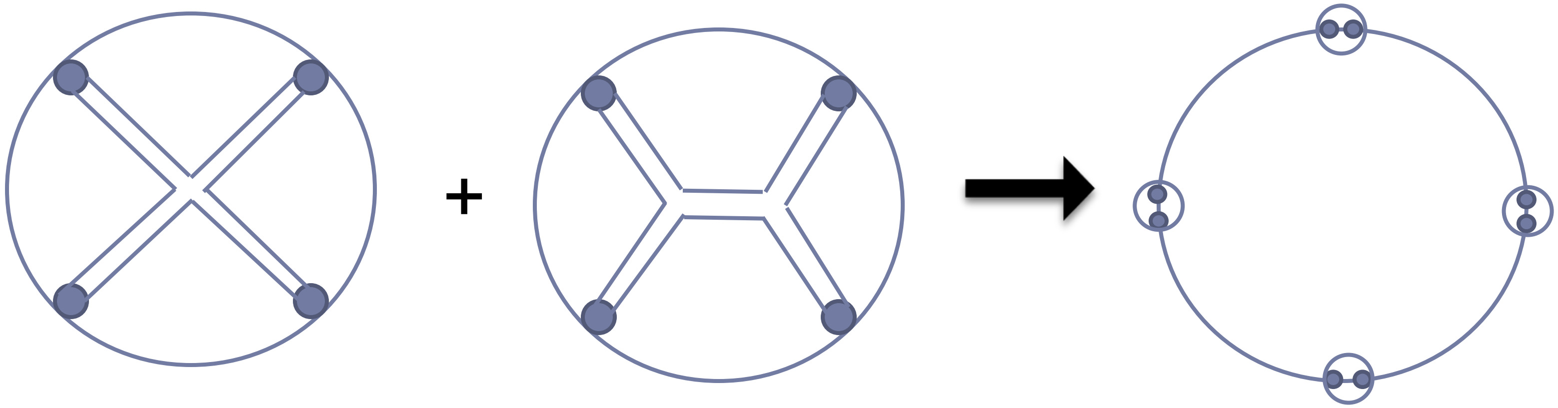
$\frac{1}{\sqrt{N}}$  $\langle \Phi(1)\Phi(2)\Phi(3)\rangle$ Collective Field Theory





Field Theory

Four-point function: $\langle \Phi(1)\Phi(2)\Phi(3)\Phi(4) \rangle$

$\frac{1}{N}$ 

Need to demonstrate

$$(x_1^\mu, x_2^\mu) \leftrightarrow AdS_4(x^\mu, z)$$

3+3 dimensions
Bi-local space-time

4 dimensions

+ Higher-Spin fields

Physical gauge:

- To establish a one-to-one relationship with the HS theory in 4d AdS space-time, we will use a physical (axial) picture
- Hamiltonian version
- Hidden gauge symmetry
- Relativistic dipole: (x_1^μ, x_2^μ)

$$L_0 = p_\mu^2 + a_\mu^\dagger a^\mu, \qquad L_1 = p^\mu a_\mu, \qquad L_{-1} = p^\mu a_\mu^\dagger$$

- Allows a single time gauge fixing

$$x_1^0 = x_2^0 = t$$

- and a Hamiltonian formulation

- The bi-local fields $\Phi_c(x,y)$ has a one-time description.
- Canonical collective field representation:

Equal-time fields

$$\Psi(t;\vec{x},\vec{y}) = \sum_a \phi^a(t,\vec{x}) \cdot \phi^a(t,\vec{y})$$

Conjugate momenta:

$$\Pi(\vec{x},\vec{y}) = -i\frac{\delta}{\delta\Psi(\vec{x},\vec{y})}$$

- The Hamiltonian is given as

$$H = 2\mathrm{Tr}(\Pi\Psi\Pi) + \frac{1}{2}\int[-\nabla_x^2\Psi(\tilde{x},\tilde{y})|_{\tilde{x}=\tilde{y}}] + \frac{N^2}{8}\mathrm{Tr}\Psi^{-1}$$

- Scaling of N: $\Psi \to N\Psi, \Pi \to \Pi/N$ and the expansion parameter is $1/N$.
- 1/N expansion

$$\Psi(x,y) = \psi_0(x,y) + \frac{1}{\sqrt{N}}\hat{\eta}(x,y), \qquad \Pi = \sqrt{N}\hat{\pi}$$

- An infinite series is generated by the expansion

$$\mathrm{Tr}\Psi^{-1} = \mathrm{Tr}\psi_0^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{N^{\frac{n}{2}}}\mathrm{Tr}(\psi_0(\hat{\eta}\psi_0)^2)$$

- The cubic vertex

$$H_3 = \frac{2}{\sqrt{N}}\mathrm{Tr}(\hat{\pi}\hat{\eta}\hat{\pi}) - \frac{1}{8\sqrt{N}}\mathrm{Tr}(\psi_0\hat{\eta}\psi_0\hat{\eta}\psi_0\hat{\eta}\psi_0)$$

- Quadratic Hamiltonian:

$$H^{(2)} = 2\mathrm{Tr}(\hat{\pi}\psi_0\hat{\pi}) + \frac{1}{8}\mathrm{Tr}(\psi_0\hat{\eta}\psi_0\hat{\eta}\psi_0)$$

is diagonalized in momentum space:

$$H^{(2)} = \frac{1}{2}\sum_{k_1 k_2} \pi_{k_1 k_2}\pi_{k_1 k_2} + \frac{1}{8}\sum_{k_1 k_2} \eta_{k_1 k_2}({\psi^0}_{k_1}^{-1} + {\psi^0}_{k_2}^{-1})^2 \eta_{k_1 k_2}$$

- Spectrum

$$\omega_{k_1,k_2} = \frac{1}{2}{\psi^0}_{k_1}^{-1} + \frac{1}{2}{\psi^0}_{k_2}^{-1} = \sqrt{{k_1}^2} + \sqrt{{k_2}^2}.$$

Equivalently:

- Quadratic Hamiltonian and momentum can be written in terms of bi-local fields as

$$H^{(2)} = \int d\vec{x} d\vec{y} \psi^\dagger(\vec{x},\vec{y})\Big(\sqrt{-\nabla_x^2} + \sqrt{-\nabla_y^2}\Big)\psi(\vec{x},\vec{y}),$$

$$P^{(2)} = \int d\vec{x} d\vec{y} \psi^\dagger(\vec{x},\vec{y})(\partial_{\vec{x}} + \partial_{\vec{y}})\psi(\vec{x},\vec{y}).$$

- Using light-cone quantization, we have:

$$P^-_{(2)} = H_{(2)} + P_{(2)} = \int dx_1^- dx_2^- dx_1 dx_2 \psi^\dagger\Big(-\frac{\nabla_1^2}{2p_1^+} - \frac{\nabla_2^2}{2p_2^+}\Big)\psi$$

Conformal group

- To establish a 1-1 map to HS AdS_4 theory, we will concentrate on the realization of the conformal group

- In light-cone quantization: $x^+ = \tau$ is the propagating time

$$\phi(x^-, x^i) = \int_0^\infty \frac{dp^+}{\sqrt{2\pi}} \frac{1}{\sqrt{2p^+}} \Big(a(p^+, x^i) e^{ip^+ x^-} + a^\dagger(p^+, x^i) e^{-ip^+ x^-} \Big),$$

(In d=3, we have only one transverse coordinate: $x^i = x$)

$$x^\mu = (x^+, x^-, x) = (\tau, x^-, x)$$

Under conformal transformation:

$$P^- : \qquad \delta a(p^+, x^i) = \frac{\partial_i^2}{2p^+} a(p^+, x^i),$$

$$M^{+-} : \qquad \delta a(p^+, x^i) = \left(t\frac{\partial_i^2}{2p^+} - i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right) a(p^+, x^i)$$

$$M^{-i} : \qquad \delta a(p^+, x^i) = \left(-\partial_i \frac{\partial}{\partial p^+} - \frac{\partial_j x^i \partial_j}{2p^+}\right) a(p^+, x^i)$$

$$D : \qquad \delta a(p^+, x^i) = \left(t\frac{\partial_i^2}{2p^+} + i\left[d_\phi + x^i\partial_i + \sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right]\right) a(p^+, x^i)$$

$$K^+ : \qquad \delta a(p^+, x^i) = \left\{t^2\frac{\partial_i^2}{2p^+} + it(d_\phi + x^i\partial_i) - \frac{1}{2}x^i x^i p^+\right\} a(p^+, x^i)$$

- and similarly for other conformal generators.

For the bi-local field (dipole):

- These induce transformations of the bi-local fields.
- In creation-annihilation form

$$A(x_1^-, x_2^-, \vec{x}_1, \vec{x}_2) = a(x_1^-, \vec{x}_1)a(x_2^-, \vec{x}_2)$$

we deduce the bi-local transformations as:

$$\delta A(1,2) = \delta a(1)a(2) + a(1)\delta a(2)$$

giving the bi-local generators:

$$G = \int dx_1^- dx_2^- dx_1 dx_2 A^\dagger \hat{g} A = \int dx_1^- dx_2^- dx_1 dx_2 A^\dagger (\hat{g}_1 + \hat{g}_2) A.$$

with $\hat{g}_1 + \hat{g}_2$ representing the two-particle "dipole" generators.

They take the form:

$$\hat{p}^- = p_1^- + p_2^- = -\left(\frac{p_1^i p_1^i}{2p_1^+} + \frac{p_2^i p_2^i}{2p_2^+}\right)$$

$$\hat{m}^{-i} = x_1^- p_1^i + x_2^- p_2^i + x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+}$$

$$\hat{d} = t\hat{p}^- + x_1^- p_1^+ + x_2^- p_2^+ + x_1^i p_1^i + x_2^i p_2^i + 2d_\phi$$

$$\hat{k}^+ = t^2 \hat{p}^- + t(x_1^i p_1^i + x_2^i p_2^i + 2d_\phi) - \frac{1}{2} x_1^i x_1^i p_1^+ - \frac{1}{2} x_2^i x_2^i p_2^+$$

with a total of 10 generators operating in the 5d dipole space:

$$(\tau; x_1^-, x_1; x_2^-, x_2)$$

II. HIGHER SPIN THEORY IN ADS_4

- A spin s field in the bulk: s-symmetric and double traceless tensor

$$h^{\mu_1\cdots\mu_s}, \quad \text{symmetric in } \mu_1,\ldots,\mu_s, \quad g_{\mu_1\mu_2}g_{\mu_3\mu_4}h^{\mu_1\cdots\mu_s}=0$$

- EOM:

$$\nabla_\rho\nabla^\rho h_{\mu_1\ldots\mu_s} - s\nabla_\rho\nabla_{\mu_1}h^\rho_{\mu_2\ldots\mu_s} + \tfrac{1}{2}s(s-1)\nabla_{\mu_1}\nabla_{\mu_2}h^\rho_{\rho\mu_3\ldots\mu_s} + \\ +2(s-1)(s+d-3)h_{\mu_1\ldots\mu_s}=0$$

- Gauge transformations:

$$\delta_\Lambda h^{\mu_1\cdots\mu_s} = \nabla^{\mu_1}\Lambda^{\mu_2\cdots\mu_s}, \quad g_{\mu_2\mu_3}\Lambda^{\mu_2\cdots\mu_s}=0$$

Fronsdal

Vasiliev ('80-'92):

- Developed an extension to include interactions.
- Fields in the HS Lie algebra:

$$\Phi(x|y,\bar{y},z,\bar{z}) = \sum \phi_{(n,m,n',m')}(x) y^n \bar{y}^m z^{n'} \bar{z}^{m'}$$

- Star product:

$$f(y,z) * g(y,z) = \int d^2u d^2v e^{u^\alpha v_\alpha} f(y+u, z+u) g(y+v, z-v),$$

- EOMs:

(1) $$d_x W + W * W = 0,$$

(2) $$d_Z W + d_x S + \{W, S\}_* = 0,$$

(3) $$d_Z S + S * S = B * K dz^2 + B * \bar{K} d\bar{z}^2,$$

(4) $$d_x B + W * B - B * \pi(W) = 0,$$

(5) $$d_Z B + S * B - B * \pi(S) = 0.$$

- Gauge transformations:

$$\delta W = d\epsilon + [W, \epsilon]_*,$$
$$\delta S = d_Z \epsilon + [S, \epsilon]_*,$$
$$\delta B = B * \pi(\epsilon) - \epsilon * B,$$

- AdS background:

$$W = W_0, \quad S = 0, \quad B = 0.$$

$$W_0 = \omega_0^L + e_0$$

$$\omega_0^L = \frac{1}{8}\frac{dx^i}{z}\left[(\sigma^{iz})_{\alpha\beta} y^\alpha y^\beta + (\sigma^{iz})_{\dot\alpha\dot\beta} \bar{y}^{\dot\alpha} \bar{y}^{\dot\beta}\right],$$
$$e_0 = \frac{1}{4}\frac{dx_\mu}{z}\sigma^\mu_{\alpha\dot\beta} y^\alpha \bar{y}^{\dot\beta}.$$

Light-cone Quantization: Metsaev [1999]

- The HS fields can be represented in Fock space using creation and annihilation operators $\alpha^A, \bar{\alpha}^A$

$$|\Phi\rangle = \Phi^{A_1 \dots A_s} \alpha^{A_1} \dots \alpha^{A_s} |0\rangle\,, \qquad \bar{\alpha}^A |0\rangle = 0\,.$$

- Restrictions to single spins is done through the constraint

$$\alpha\bar{\alpha}|\Phi\rangle = s|\Phi\rangle\,, \quad \alpha\bar{\alpha} \equiv \alpha^A \bar{\alpha}^A\,.$$

- With the additional double traceless condition

$$(\bar{\alpha}^2)^2 |\Phi\rangle = 0\,, \qquad \bar{\alpha}^2 \equiv \bar{\alpha}^A \bar{\alpha}^A\,.$$

- Gauge invariant EOM

$$\Big([\bar{\alpha}D, \alpha D] - \alpha D\bar{\alpha}D + \frac{1}{2}(\alpha D)^2\bar{\alpha}^2 - 2\alpha^2\bar{\alpha}^2 + 2(2s + d - 3)\Big)|\Phi\rangle = 0$$

- Gauge transformation

$$\delta|\Phi\rangle = \alpha D|\Lambda\rangle \qquad \bar{\alpha}^2|\Lambda\rangle = 0\,.$$

- The Lorentz covariant derivative

$$D_\mu \equiv \partial_\mu + \frac{1}{2}\omega_\mu^{AB}M^{AB}\,, \qquad M^{AB} = \alpha^A\bar{\alpha}^B - \alpha^B\bar{\alpha}^A\,.$$

Fixing the Light-cone gauge

- In four dimensions, the only physical states are the ±s helicity states.
- Staring from the covariant notation

$$|\Phi\rangle = \sum_{s=1}^{\infty} \Phi^{\mu_1 \dots \mu_s} a^{\dagger}_{\mu_1} \dots a^{\dagger}_{\mu_s} |0\rangle$$

- Step 1: drop the oscillators $a^{\pm} = a^0 \pm a^d$ and keep only the transverse oscillators $a^I, a^{\dagger J}$
- Step 2: further constraint equation

$$T|\Phi\rangle = 0, \qquad T = a^I a^I$$

- Complex oscillators

$$\alpha = \frac{1}{\sqrt{2}}(a_1 + ia_2), \qquad \alpha^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger + ia_2^\dagger),$$
$$\bar{\alpha} = \frac{1}{\sqrt{2}}(a_1 - ia_2), \qquad \bar{\alpha}^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger - ia_2^\dagger),$$

- Expansion:

$$|\Phi\rangle = \sum_{\lambda=1}^{\infty}\Big(\Phi_{(\lambda)}(\bar{\alpha}^\dagger)^\lambda + \bar{\Phi}_{(\lambda)}(\alpha^\dagger)^\lambda\Big)|0\rangle$$

- The constraint equation

$$T|\Phi\rangle = 0, \qquad T = \bar{\alpha}\alpha.$$

- The spin matrix

$$M = \alpha^\dagger\bar{\alpha} - \bar{\alpha}^\dagger\alpha$$

Light-cone form of EOM

- Through the metric: $$ds^2 = \frac{2dtdx^- + dx^2 + dz^2}{z^2}$$
- Light-cone equations of motion

$$\left(z^2(2\partial_+\partial_- + \partial_I^2) + \frac{1}{2}M_{ij}^2 - \frac{(d-4)(d-6)}{4} \right) |\phi\rangle = 0$$

- From which we deduce the generator

$$P^- = \partial_+ = -\frac{\partial_I^2}{2\partial_-} + \frac{1}{2z^2\partial_-}\left(-\frac{1}{2}M_{ij}^2 + \frac{(d-4)(d-6)}{4} \right)$$

- Other generators of the space-time SO(2,3) group can be evaluated similarly.

First quantized generators

- In four dimensions, the only non-vanishing spin matrix is M^{xz}
- One can represent $\alpha = e^{i\theta},\ \bar{\alpha} = e^{-i\theta}.$ so that the operator becomes

$$M^{xz} \longrightarrow \frac{\partial}{\partial\theta}$$

- The physical field

Extra AdS coordinate

$$\Phi(x^+, x^-, x, z; \theta)$$

HS coordinate

- Symmetry generators

$$G = \int dx^- dx dz d\theta\ \bar{\Phi}\hat{g}\Phi$$

Explicitly:

$$\hat{p}^{-} = -\frac{p^x p^x + p^z p^z}{2p^+},$$

$$\hat{m}^{-x} = x^- p^x - x\hat{p}^- + \frac{p^\theta p^z}{p^+}$$

$$\hat{d} = t\hat{p}^- + x^- p^+ + xp^x + zp^z + d_a$$

$$\hat{k}^+ = t^2\hat{p}^- + t(xp^x + zp^z + d_a) - \frac{1}{2}(x^2 + z^2)p^+$$

……

III. ONE-TO-ONE MAP:

- CFT$_3$: collective bi-local fields AdS$_4$: higher spin fields

$$\Psi(x^+; (x_1^-, x_1), (x_2^-, x_2)) \Longleftrightarrow \Phi(x^+; x^-, x, z; \theta)$$

- Same number of dimensions

  1+2+2 = 1+3+1

- Representation of the conformal group SO(2,3)
- Clear from analysis of the two representations that one does not have a coordinate transformation

Solution: canonical transformation

- Identifying the generators of the dipole with the generators of HS: gives 10 equations of 2×4=8 canonical variables

$$x^- = \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+},$$

$$p^+ = p_1^+ + p_2^+,$$

$$x = \frac{x_1 p_1^+ + x_2 p_2^+}{p_1^+ + p_2^+},$$

$$p^x = p_1 + p_2.$$

$$z = \frac{(x_1 - x_2)\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+},$$

$$p^z = \sqrt{\frac{p_2^+}{p_1^+}}p_1 - \sqrt{\frac{p_1^+}{p_2^+}}p_2,$$

$$\theta = 2\arctan\sqrt{\frac{p_2^+}{p_1^+}}.$$

$$p^\theta = \sqrt{p_1^+ p_2^+}(x_1^- - x_2^-) + \frac{x_1 - x_2}{2}\Big(\sqrt{\frac{p_2^+}{p_1^+}}p_1 + \sqrt{\frac{p_1^+}{p_2^+}}p_2\Big).$$

Poisson brackets

- We have the AdS_4 canonical variables in terms of the dipole canonical variables (of CFT_3).
- A consistency check on the correctness of the map: the Poisson brackets take the canonical form.
- Assuming

$$\{x_i^-, p_i^+\} = \{x_i, p_i^x\} = 1$$

we verify that

$$\{x^-, p^+\} = \{x, p^x\} = \{z, p^z\} = \{\theta, p^\theta\} = 1$$

- We have a successful reconstruction of AdS space-time from the bi-local one.

Question

- Since there is no transformation between coordinates of AdS_4 and bi-local coordinates emerging from CFT_3, one has the question how the respective fields are related?

$$\Phi(x^\mu, z; \alpha) \leftrightarrow \Psi(x^+; (x_1^-, x_1), (x_2^-, x_2))$$

- The relationship between fields:

✓ From the canonical transformation we can deduce that there exists a (non-local) kernel K (AdS/dipole):

$$\Phi(x^\mu, z; \alpha) = \int K(x^\mu, z; \alpha | x_\mu^1, x_\mu^2) \Psi_c(x_1^\mu, x_2^\mu) d^3x_1 d^3x_2$$

✓ Construction of the kernel is possible

IV. The Collective Dipole

- Two relativistic particle:

$$S = \int d\tau m \sqrt{-\dot{x}_1^2(\tau)} + \int d\tau' m \sqrt{-\dot{x}_2^2(\tau')}$$

- Equations of motion:

$$p_1^2 + m^2 = 0 \qquad\qquad p_2^2 + m^2 = 0$$

- Center-of-mass frame:

$$P^2 + p^2 + 4m^2 = 0 \qquad\qquad P \cdot p = 0$$

- Elimination of relative time: gauge fix $\updownarrow$

$$P \cdot x = 0$$

Solving the constraints:

- Canonical transformation

$$P_\mu = P_\mu,$$

$$X^\mu = \tilde{X}^\mu + u^L b^{\mu s}\pi_s - \frac{\pi_L}{P^2} b_s{}^\mu u^s - u^r \pi_s b_r{}^\nu \frac{\partial b_\nu{}^s}{\partial P_\mu} + \frac{u^L \pi_L}{P^2} P^\mu,$$

$$p_\mu = \frac{P_\mu}{P^2}\pi_L + b_\mu{}^r \pi_r,$$

$$x^\mu = P^\mu u^L + b_r{}^\mu u^r$$

- Constraints lead to $u^L = \pi_L = 0$.
- One time: $\tilde{X}^0 = t$
- Hamiltonian

$$P_0 = \sqrt{\vec{P}^2 + p^2 + 4m^2}$$

V. CONSTUCTION OF THE INTERTWINING MAP

- General integral transformation:

$$\Psi(U) = \int dX\,dP\; e^{iF(U,P) - iP\cdot X}\,\Phi(X)$$

where the generating function *F(U,P)* satisfies

$$\frac{\partial F}{\partial U} = \Pi(U,P), \qquad \frac{\partial F}{\partial P} = X(U,P).$$

- Saddle points approximation

$$\frac{\partial F}{\partial P} - X = 0 \Rightarrow P = P^*(X,U).$$

$$\Psi(U) = \int dX\, e^{iF(U,P^*(X,U)) - iP^*(X,U)\cdot X}\,\Phi(X)$$

The Liouville example:

$$\left[\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial \varphi^2} - e^{-\varphi}\right)\right]\Psi(t,\varphi) = 0 \quad \Longleftarrow \quad \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \sigma^2}\right]\Phi(t,\sigma) = 0$$

- Canonical transformation:

$$\begin{aligned} \sinh\sigma &= -p_\sigma e^{-\varphi}, \\ \pi_\varphi &= e^{\varphi}\sqrt{1 + p_\sigma^2 e^{-2\varphi}}. \end{aligned}$$

- Generating function:

$$F(\varphi, p_\sigma) = -p_\sigma \ln\left(p_\sigma e^{-\varphi} + \sqrt{p_\sigma^2 e^{-2\varphi} + 1}\right) + e^{\varphi}\sqrt{p_\sigma^2 e^{-2\varphi} + 1}$$

- Integral transformation:

$$\Psi(t,\varphi) = \int d\sigma \exp(e^{\varphi}\cosh\sigma)\Phi(t,\sigma)$$

From bi-local field to HS field:

- In the dual (momentum) space: (p_1^+, p_2^+, p_1, p_2), one observes the transformation takes the form of a point transformation:

$$
\begin{aligned}
p^+ &= p_1^+ + p_2^+ , \\
p^x &= p_1 + p_2 , \\
p^z &= p_1\sqrt{p_2^+/p_1^+} - p_2\sqrt{p_1^+/p_2^+} , \\
\theta &= 2\arctan\sqrt{p_2^+/p_1^+} ,
\end{aligned}
$$

- Consequently perform a Fourier transform:

$$
\tilde{\Phi}(p_1^+, p_2^+, p_1, p_2) = \int dx_1^- dx_2^- dx_1 dx_2 e^{-i(x_1^- p_1^+ + x_2^- p_2^+ + x_1 p_1 + x_2 p_2)} \Phi(x_1^-, x_2^-, x_1, x_2)
$$

- Changing to AdS variables using an inverse transform gives the AdS higher-spin field in terms of the bi-local one:

$$
\begin{aligned}
\Psi(x^-, x, z, \theta) \quad = \quad & \int dp^+ dp^x dp^z e^{i(x^- p^+ + x p^x + z p^z)} \\
& \int dp_1^+ dp_2^+ dp_1 dp_2 \delta(p_1^+ + p_2^+ - p^+) \delta(p_1 + p_2 - p^x) \\
& \delta\left(p_1 \sqrt{p_2^+/p_1^+} - p_2 \sqrt{p_1^+/p_2^+} - p^z\right) \\
& \delta(2 \arctan \sqrt{p_2^+/p_1^+} - \theta) \tilde{\Phi}(p_1^+, p_2^+, p_1, p_2)
\end{aligned}
$$

Checking the z=0 projection

- The bilocal field gives a strong off-shell operator re-construction of the HS field: $\Phi = W|_{\text{gauge}}$
- One check on our identification of the extra AdS coordinate z is the evaluation of the z=0 limit
- We expect that they reduce to the conformal operators
- At z=0:

$$\begin{aligned}\Phi(x^+, x^-, x, \theta) &= \int dp_1^+ dp_2^+ e^{ix^-(p_1^+ + p_2^+)} \\ &\quad \delta(\theta - 2\tan^{-1}\sqrt{p_2^+/p_1^+})\tilde{\psi}(p_1^+, p_2^+, x, x)\end{aligned}$$

- Expanding the delta function in Fourier series, one has the binomial expansion

$$(\sqrt{p_1^+} - i\sqrt{p_2^+})^{2s} = \frac{(-1)^k (2s)!}{(2k)!(2s-2k)!}(p_1^+)^k (p_2^+)^{s-k}$$

- The conformal operators for a fixed spin s is

$$\mathcal{O}^s = \sum_{k=0}^{s} \frac{(-1)^k \Gamma(s+1/2)\Gamma(s+1/2)}{k!(s-k)!\Gamma(s-k+1/2)\Gamma(k+1/2)} (\partial_+)^k \varphi (\partial_+)^{s-k} \varphi$$

- The expansion coefficients agree up to an overall normalization

$$\frac{(2s)!}{(2k)!(2s-2k)!} = \frac{s!\ \Gamma(s+1/2)\Gamma(1/2)}{k!(s-k)!\ \Gamma(s-k+1/2)\Gamma(k+1/2)}$$

VI. Covariant gauge: Map

- Vasiliev's gauge fields W_μ, S_α, B are functions of both space-time and two sets of spinors

$$F(X_{\rm ads}; y, \bar{y}; z, \bar{z})$$

- One possible gauge leads to comparison with Fronsdal's formulation

$$\Phi(\underbrace{X_{\rm ads}}_{4}; \underbrace{y, \bar{y}}_{4})$$

- Symmetric, double-traceless tensor fields:

$$h_{\mu_1 \ldots \mu_s} \qquad g^{\mu_1\mu_2} g^{\mu_3\mu_4} h_{\mu_1\mu_2\mu_3\mu_4\ldots\mu_s} = 0$$

- Covariant gauge conditions:

$$\nabla^\mu h_{\mu\mu_2\ldots\mu_s} = 0 \qquad g^{\mu_1\mu_2} h_{\mu_1\mu_2\ldots\mu_s} = 0$$

- Embedding AdS_4 into R^5:

$$x^\mu \to y^\alpha \qquad y^\alpha y_\alpha = -1, \quad \alpha = 0, 1, 2, 3, 5$$

- Higher-spin fields:

$$h_{\mu\ldots} = y^\alpha_\mu \cdots k_{\alpha\ldots} \qquad\qquad y^\alpha_\mu = \partial y^\alpha / \partial x^\mu$$

- Synthesis of all integer spins

$$K(y, z) = \sum_s z^{\alpha_1} \cdots z^{\alpha_s} k_{\alpha_1 \cdots \alpha_s}(y)$$

- Summary of all conditions:

	$k_{\alpha_1 \cdots \alpha_s}(y)$	$K(y,z)$
transversality	$y \cdot k = 0$	$y \cdot \partial_z K = 0$
gauge condition	$\partial \cdot k = 0$	$\partial_y \cdot \partial_z K = 0$
traceless	$k' = 0$	$\partial_z^2 K = 0$
~~$y^2 = -1$~~	$(y \cdot \partial_y + s + 1)k = 0$	$(y \cdot \partial_y + z \cdot \partial_z + 1)K = 0$

Mapping to symmetric representation:

$(y \cdot \partial_y + z \cdot \partial_z + 1)K = 0$	$(p \cdot \partial_p + 1/2)\Phi = 0$
$y \cdot \partial_z K = 0$	$(q \cdot \partial_q + 1/2)\Phi = 0$
EOM $\partial_y^2 K = 0$	$\partial_p^2 \Phi = 0$
$\partial_z^2 K = 0$	$\partial_q^2 \Phi = 0$
$\partial_y \cdot \partial_z K = 0$	$\partial_p \cdot \partial_q \Phi = 0$ EOM

First-class constraints

This gives a 3+3 dimensional representation.

One likes to extend this to the nonlinear level.

VII. SYMMETRIC GAUGE

- Bi-local field theory is symmetric:

$$\Psi(x_\mu^1, x_\mu^2)$$

- There is a symmetric gauge in Vasiliev's theory:

$$F(y, \bar{y}; z, \bar{z})$$

- Solving the zero curvature equation using the pure gauge solution

$$W_\mu = g^{-1} * \partial_\mu g$$

- Fixing the gauge g(x)=1, one ends up with the equations

$$d_Z S + S * S = B * (K dz^2 + \bar{K} d\bar{z}^2)$$
$$d_Z B + S * B - B * \pi(S) = 0$$

W=0 gauge

- In components, one has five independent equations

$$F_{z^1\bar{z}^2} = \partial_1\bar{S}_2 - \bar{\partial}_2 S_1 + [S_1, \bar{S}_2]_* = 0$$

$$F_{z^2\bar{z}^1} = \partial_2\bar{S}_1 - \bar{\partial}_1 S_2 + [S_2, \bar{S}_1]_* = 0$$

$$F_{z^1\bar{z}^1} = \partial_1\bar{S}_1 - \bar{\partial}_1 S_1 + [S_1, \bar{S}_1]_* = 0$$

$$F_{z^2\bar{z}^2} = \partial_2\bar{S}_2 - \bar{\partial}_2 S_2 + [S_2, \bar{S}_2]_* = 0$$

$$F_{z^1 z^2} * K = F_{\bar{z}^1\bar{z}^2} * \bar{K}$$

Reality condition for the B field

Analog with self-dual Yang-Mills

HS	SDYM
$F_{z^1\bar{z}^2}=0$	$F_{yz}=0$
$F_{z^2\bar{z}^1}=0$	$F_{\bar{y}\bar{z}}=0$
$F_{z^1\bar{z}^1}+F_{z^2\bar{z}^2}=0$	$F_{y\bar{y}}+F_{z\bar{z}}=0$
$F_{z^1\bar{z}^1}-F_{z^2\bar{z}^2}=0$	
$F_{z^1z^2}*K=F_{\bar{z}^1\bar{z}^2}*\bar{K}$	

- Coordinates: $z^1=y,\quad \bar{z}^1=\bar{y},\quad z^2=\bar{z},\quad \bar{z}^2=z$

An ansatz:

- Using the ansatz

$$S_1 = M^{-1} * \partial_1 M, \qquad S_2 = \bar{M}^{-1} * \partial_2 \bar{M},$$
$$\bar{S}_1 = \bar{M}^{-1} * \bar{\partial}_1 \bar{M}, \qquad \bar{S}_2 = M^{-1} * \bar{\partial}_2 M.$$

- F_{12} and F_{21} are solved, F_{11} and F_{22} become

$$\bar{\partial}_1 (J^{-1} * \partial_1 J) = 0 \qquad \text{(I a)}$$
$$\partial_2 (J^{-1} * \bar{\partial}_2 J) = 0 \qquad \text{(I b)}$$

where $J \equiv M * \bar{M}^{-1}$ is a (residual) gauge invariant quantity.

- The last equation becomes

$$\partial_2 (J^{-1} * \partial_1 J) * K + \bar{\partial}_1 (J^{-1} * \bar{\partial}_2 J) * \bar{K} = 0 \qquad \text{(II)}$$

- We now have equations for a single scalar field:

$$J(y, \bar{y}; z, \bar{z})$$

- Equation (Ia, Ib) can be thought of as constraints giving a reduction:

$$4+4 \longrightarrow 3+3$$

- Equation (II) represents an equation of motion
- One can expect that an Action can be written down for this system
- Closest in form to the covariant version of collective field equation of motion

VIII.Related Questions/Work

- Construction of a strong operator AdS HS theory correspondence:

$$\Phi = W\big|_{\text{gauge}}$$

- Intertwining map: generating the AdS_4 as an Emergent space-time
- Action: $S = \mathrm{Tr}\left((\partial_1^2 + \partial_2^2)\Phi + \ln\Phi\right)$

  and the Hamiltonian:

$$H = \mathrm{Tr}\left(\frac{1}{2}\Pi\Psi\Pi + (\nabla_1^2 + \nabla_2^2)\Psi + \frac{N}{\Psi}\right)$$

  represents a covariant and time-like gauge fixing of Vasiliev's theory, respectively.

- Interesting question of integrability, higher conservation laws…
- Question of quantum corrections (loop)
- It is relevant to extend our construction to other cases, for example $AdS_3 \longrightarrow CFT_2$ HS studied by

Gaberdiel, Gopakumar, Hartman,
Henneaux, Rey
Castro, Maloney

- S_N symmetric CFT_2 considered by

AJ, Mihailescu, Ramgoolam '99